

## Lecture 13 (3/21/16)

**Theorem 4.9.** Suppose assumptions of Theorem 4.8 are satisfied and  $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$ ,  $\forall t \geq 0$  &  $\forall x \in D$  where  $W_3(x)$  is a continuous, positive definite on  $D$ . Then  $x=0$  is u.a.s.

Moreover, if  $r, c$  chosen s.t.  $B_r \subset D$  &  $c < \min_{\|x\|=r} W_1(x)$ , then every trajectory starting in  $G_2 = \{x \in B_r \mid W_2(x) < c\}$  satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0), \quad \forall t \geq t_0 \geq 0$$

$B \in KL$ . If  $D = \mathbb{R}^n$  and  $W_1$  is radially unbounded  $\Rightarrow x=0$  is g.u.a.s.

**Proof.** Continuing with the proof of Thm 4.8, trajectories starting in  $G_2 = \{W_2 < c\}$  stays in  $G_1 = \{W_1 < c\}$ ,  $\forall t \geq t_0$ . By lemma 4.3.  $\exists \alpha_3 \in K$  s.t.  $\dot{V} \leq -W_3(x) \leq -\alpha_3(\|x\|)$

$$\text{and } V \leq \alpha_2(\|x\|) \Leftrightarrow \alpha_2^{-1}(V) \leq \|x\| \Leftrightarrow \alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$$

Let  $\alpha_3 \circ \alpha_2^{-1} := \alpha$ . So  $\alpha(V) \leq \alpha_3(\|x\|)$  or  $-\alpha_3(\|x\|) \leq -\alpha(V)$ . So  $\dot{V} \leq -\alpha(V)$ .

Let  $y(t)$  satisfy  $\dot{y} = -\alpha(y)$ ,  $y(t_0) = V(t_0, x(t_0)) \geq 0$ .

By lemma 3.4 (comparison lemma)  $V(t, x(t)) \leq y(t)$ ,  $\forall t \geq 0$ .

By lemma 4.4,  $\exists \epsilon(r, s) \in KL$  on  $[0, r] \times [0, \infty)$  s.t.

$$V(t, x(t)) \leq \epsilon(V(t_0, x(t_0)), t-t_0) \quad \forall V(t_0, x(t_0)) \in [0, c]$$

Therefore, any solution starting in  $G_2$  satisfies:

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\epsilon(V(t_0, x(t_0)), t-t_0)) \\ &\leq \alpha_1^{-1}(\epsilon(\alpha_2(\|x(t_0)\|)), t-t_0) = \beta(\|x(t_0)\|, t-t_0), \quad \beta \in KL \end{aligned} \quad (*)$$

$\therefore x=0$  is u.a.s. (lemma 4.5)

$D = \mathbb{R}^n \Rightarrow \alpha_i$  defined on  $[0, \infty)$   $\Rightarrow \alpha$  & hence  $\beta$  are independent of  $c$ .

As  $W_1$  is radially unbounded,  $c$  can be chosen arbitrary large to include any initial state in  $G_2$ . Thus (\*) holds for any initial state.

$\therefore x=0$  is g.u.a.s.

## Exponential stability

$x=0$  is exponentially stable if  $\exists c > 0, k > 0, \lambda > 0$  s.t.

$$\|x(t)\| \leq K \|x(t_0)\| e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c$$

and globally exponentially stable if true for any initial condition  $x(t_0)$ .

→ special case of (g) u.a.s. when class KL function is  $\beta(rs) = kr e^{-\lambda s}$

### Theorem 4.10 (exponential stability theorem)

Let  $x=0$  be an eq. pt. for  $\dot{x} = f(t, x)$ . Let  $V: [t_0, \infty) \times D \rightarrow \mathbb{R}$  be  $C^1$  s.t.

$$K_1 \|x\|^{\alpha} \leq V(t, x) \leq K_2 \|x\|^{\alpha}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -K_3 \|x\|^{\alpha}$$

$\forall t \geq t_0$  &  $\forall x \in D$ , where  $K_i$ 's are positive constants. Then  $x=0$  is e.s.

If assumptions hold globally, then  $x=0$  is g.e.s.

Proof. Can show that  $\dot{V} \leq -\frac{K_3}{K_2} V$ . By comparison lemma 3.4.

$$V(t, x(t)) \leq V(t_0, x(t_0)) e^{-\frac{K_3}{K_2}(t-t_0)}$$

using this in  $\|x(t)\| \leq (\frac{V}{K_1})^{\frac{1}{\alpha}}$  and using  $\|V(t_0, x(t_0))\| \leq K_2 \|x(t_0)\|^{\alpha}$ , we have:

$$\|x(t)\| \leq \left(\frac{K_2}{K_1}\right)^{\frac{1}{\alpha}} \|x(t_0)\| e^{-\frac{K_3}{K_2\alpha}(t-t_0)}$$

### Linear time varying system $\dot{x} = A(t)x$

We cannot look at eigenvalues of  $A(t)$  at each fixed  $t$  to prove stability.

Example.  $A(t) = \begin{pmatrix} -1 + \frac{3}{2} \cos^2 t & 1 - \frac{3}{2} \sin t \cos t \\ -1 - \frac{3}{2} \cos t \sin t & -1 + \frac{3}{2} \sin^2 t \end{pmatrix}$

For each  $t$  eigenvalues are  $\frac{-1 \pm i\sqrt{7}}{4}$ . But  $\phi(t, 0) = \begin{pmatrix} e^{t/2} \cos t & e^{-t} \sin t \\ -e^{t/2} \sin t & e^t \cos t \end{pmatrix}$   
 $\therefore \exists$  states  $x(0)$  arbitrary close to origin for which solution is unbounded and escapes. e.g.  $x(0) = \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} \Rightarrow x(t) = \epsilon e^{t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ .

Theorem 4.11. The eq. pt  $\dot{x}=0$  of  $\dot{x}=A(t)x$  is g.u.a.s. iff the state transition matrix satisfy the inequality:

$$\|\phi(t, t_0)\| \leq K e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 > 0 \text{ for some positive constants } k, \lambda.$$

where  $\phi(t, t_0)$  is state transition matrix:  $x(t) = \phi(t, t_0) x(t_0)$ .

- This theorem shows that for linear systems, u.a.s & e.s. are equivalent.
- Although this might not be a useful criterion to show u.a.s. in linear systems since to find  $\phi(t, t_0)$  one need to solve the equation, one can use the bound to show that there exists a Lyapunov function for the system. (see Theorem 4.12. below)

Proof of Thm 4.11.  $x(t)$  depends on  $x(t_0)$  linearly  $\Rightarrow$  u.a.s implies g.u.a.s.

sufficiency:  $\|x(t)\| \leq \|\phi(t, t_0)\| \|x(t_0)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)}$

necessity:

### sufficient condition for stability

Suppose  $A(t)$  is continuous  $\forall t \geq 0$ . Suppose  $\exists$  a  $C'$ , positive definite matrix  $P(t)$ , i.e.  $0 < C_1 I \leq P(t) \leq C_2 I$ ,  $\forall t \geq 0$

which satisfies  $-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$  where  $Q(t) \geq C_3 I > 0$  (continuous & positive definite). Then  $X=0$  is g.e.s.

**Proof.** Let  $V(t, X) = X^T P(t) X$ . Then  $C_1 \|X\|_2^2 \leq V(t, X) \leq C_2 \|X\|_2^2$  and

$$\begin{aligned}\dot{V} &= X^T \dot{P}X + X^T P \dot{X} + \dot{X}^T P X = X^T (\dot{P} + PA + A^T P) X \\ &= -X^T Q X \leq -C_3 \|X\|_2^2\end{aligned}$$

By Thm 4.10. (with  $\alpha=2$ ) get g.e.s.

### Converse theorem for linear time varying systems

**Theorem 4.12.** Let  $X=0$  be the e.s. eq.pt. of  $\dot{X}=A(t)X$ . Suppose

- $A(t)$ : continuous & bounded and let
- $Q(t)$ : continuous & bounded & positive definite

Then  $\exists C'$ , bounded, positive definite matrix  $P(t)$  that satisfies

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

Hence  $V(t, X) = X^T P(t) X$  is a Lyapunov function for the system that satisfies the condition of the e.s theorem 4.10.

**Proof.** Use  $P(t) = \int_t^\infty \phi^T(\tau, t) Q(\tau) \phi(\tau, t) d\tau$

will show the proof for non-linear system which is a generalization of this.

Note that when  $A(t) = A$  is time invariant,  $\phi(\tau, t) = e^{(t-\tau)A}$

and hence

$$P(t) = \int_t^\infty e^{(t-\tau)A^T} Q(\tau) e^{(\tau-t)A} d\tau = \int_0^\infty e^{\tau A^T} Q(\tau) e^{\tau A} d\tau$$

which is independent of  $t$  and we showed that it is the unique solution of Lyapunov equation & constructed a Lyapunov function from that.

## Linearization of non-autonomous systems

Theorem 4.13. Let  $x=0$  be an eq. pt. for the nonlinear system  $\dot{x}=f(t,x)$  where  $f: [0, \infty) \times D \rightarrow \mathbb{R}^n$  is  $C^1$ ,  $D = \{x \in \mathbb{R}^n \mid \|x\|_2 < r\}$  &  $\frac{\partial f}{\partial x}$  is bounded and Lipschitz on  $D$ , uniformly in  $t$ . Let  $A(t) = \frac{\partial f}{\partial x}(t, x)|_{x=0}$ . Then  $x=0$  is an e.s. eq.pt. of  $\dot{x}=f(t,x)$  if (P) it is an e.s. eq.pt for  $\dot{x}=A(t)x$ .

will show in Theorem 4.15

Proof is similar to autonomous case using converse theorem for time varying linear systems to get a Lyapunov function and take  $V$  along perturbed nonlinear dynamics.

Converse theorem for e.s. eq.pt.

Theorem 4.14. Let  $x=0$  be eq.pt for  $\dot{x}=f(t,x)$ , where  $f: [0, \infty) \times B_r \rightarrow \mathbb{R}^n$  is  $C^1$  &  $\frac{\partial f}{\partial x}$  is bounded in  $D$ , uniformly in  $t$ . Let  $k, \lambda, r_0$  be positive constants with  $r_0 < r/k$ . Let  $D_0 = B_{r_0}$ . Assume that trajectories satisfy  $\|x(t)\| \leq K \|x(t_0)\| e^{-\lambda(t-t_0)}, \forall x(t_0) \in D_0, t \geq t_0 \geq 0$ .

Then  $\exists V: [0, \infty) \times D_0 \rightarrow \mathbb{R}$  that satisfies

$$\begin{aligned} c_1 \|x\|^2 &\leq V(t, x) \leq c_2 \|x\|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -c_3 \|x\|^2 \quad \text{any norm} \\ \|\frac{\partial V}{\partial x}\| &\leq c_4 \|x\| \end{aligned}$$

for some positive constants  $c_1, \dots, c_4$ .

If  $r=\infty$  &  $x=0$  is g.e.s. then  $D_0 = \mathbb{R}^n$ .

If system is autonomous,  $V$  can be chosen independent of  $t$ .

Proof. For 2-norms. Let  $\phi(\tau; t, x)$  be solution that starts at  $(t, x)$ . Let

$$V(t, x) = \int_t^{t+\delta} \phi^\top(\tau; t, x) \phi(\tau; t, x) d\tau \quad \delta > 0 \text{ TBD}$$

$$\begin{aligned} \textcircled{1} \quad V(t, x) &= \int_t^{t+\delta} \|\phi(\tau, t, x)\|_2^2 d\tau \leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)} d\tau \|x\|_2^2 \\ &= \underbrace{\frac{k^2}{2\lambda} (1 - e^{-2\lambda\delta})}_{C_2} \|x\|_2^2 \end{aligned}$$

$$\textcircled{2} \quad \left\| \frac{\partial f}{\partial x}(t, x) \right\|_2 \leq L, \quad \forall x \in B_r \Rightarrow \|f(t, x)\|_2 \leq L \|x\|_2$$

$$\stackrel{\text{Exe 3.17}}{\Rightarrow} \|\phi(\tau, t, x)\|_2 \geq \|x\|_2 e^{-L(\tau-t)}$$

$$\text{Therefore, } V(t, x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau \|x\|_2^2 = \underbrace{\frac{1}{2L} (1 - e^{-2L\delta})}_{C_1} \|x\|_2^2$$

\textcircled{3} To calculate the derivative of \$V\$ along the trajectories, define the "sensitivity functions" \$\phi\_t = \frac{\partial}{\partial t} \phi\$ & \$\phi\_x = \frac{\partial}{\partial x} \phi\$. Then

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &= \left\{ \phi^T(t+\delta, t, x) \phi(t+\delta, t, x) - \phi^T(t, t, x) \phi(t, t, x) \right\} \\ &\quad + \left\{ 2 \int_t^{t+\delta} \phi^T(\tau, t, x) [\phi_t(\tau, t, x) + \phi_x(\tau, t, x) f(t, x)] d\tau \right\} \end{aligned}$$

$$\stackrel{\text{Exe 3.30}}{\longrightarrow} \phi_t(\tau, t, x) + \phi_x(\tau, t, x) f(t, x) = 0 \quad \forall \tau \geq t$$

$$\text{Therefore, } \dot{V} \leq -(1 - k^2 e^{-2\lambda\delta}) \|x\|_2^2 \leq \underbrace{-\frac{1}{2}}_{C_3} \|x\|_2^2 \quad \text{when } \delta = \frac{\ln 2k^2}{2\lambda}$$

\textcircled{4} \$\phi\_x\$ satisfies the sensitivity equation

$$\frac{\partial}{\partial \tau} \phi_x(\tau, t, x) = \frac{\partial f}{\partial x}(\tau, \phi(\tau, t, x)) \phi_x, \quad \phi_x(t, t, x) = I$$

$$\phi_\tau = f(\tau, \phi(\tau)), \quad \phi(t, t, x) = x \quad \forall x$$

$$\left\| \frac{\partial f}{\partial x} \right\| \leq L \stackrel{\text{Exe 3.17}}{\longrightarrow} \|\phi_x(\tau, t, x)\|_2 \leq e^{L(\tau-t)}$$

$$\text{Therefore, } \left\| \frac{\partial V}{\partial x} \right\|_2 = \left\| \int_t^{t+\delta} 2\phi^T \phi_x d\tau \right\|_2 \leq \int_t^{t+\delta} 2\|\phi\|_2 \|\phi_x\|_2 d\tau$$

$$\left\| \frac{\partial V}{\partial X} \right\|_2 \leq \int_t^{t+8} 2K e^{-\lambda(\tau-t)} e^{L(\tau-t)} \|X\|_2 d\tau = \underbrace{\frac{2K}{\lambda-L} (1 - e^{-(\lambda-L)8})}_{C_4} \|X\|_2^2$$

Autonomous System:  $\phi(\tau; t, x) = \psi(\tau-t; x)$

$$\Rightarrow V(t, x) = \int_t^{t+8} \psi^T(\tau-t, x) \psi(\tau-t, x) d\tau = \int_0^8 \psi^T(s, x) \psi(s, x) ds \quad (\text{independent of } t)$$

□

Theorem 4.15. Replace if with iff in Theorem 4.13.

Proof. "only if" write linear system as

$$\dot{x} = f(t, x) - \underbrace{[f(t, x) - A(t)x]}_{g(t, x)}, \quad \|g(t, x)\|_2 \leq L \|x\|_2 \quad \forall x \in D, \forall t \geq 0$$

Since  $x=0$  is e.s. for  $\dot{x}=f(t, x)$ ,  $\exists K, \lambda, c > 0$  st.

$$\|x(t)\|_2 \leq K \|x(t_0)\|_2 e^{-\lambda(t-t_0)} \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\|_2 < c$$

Choose  $r_0 < \min \{c, r_K\}$ . Then all conditions of Theorem 4.14 are satisfied. Use that  $V(t, x)$  as candidate for linear system

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) - \frac{\partial V}{\partial x} g(t, x) \\ &\leq -c_3 \|x\|_2^2 + c_4 L \|x\|_2^3 \end{aligned}$$

choose

$$f < \min \left\{ r_0, \frac{c_3}{c_4 L} \right\} \quad < 0 \quad \forall \|x\|_2 < \rho$$

$\therefore x=0$  is e.s. for linear system.

Example  $\dot{x} = -x^3$   $x=0$  is a.s. ( $V(x) = \frac{1}{2}x^2$ ) but is not e.s.  
(look at linearization)

